

# THE STRUCTURE OF SMOOTH ALGEBRAS IN KAPRANOV'S FRAMEWORK FOR NONCOMMUTATIVE GEOMETRY.

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ABSTRACT. In Kapranov, M. *Noncommutative geometry based on commutator expansions*, J. reine angew. Math **505** (1998) 73-118, a theory of noncommutative algebraic varieties was proposed. Here we prove a structure theorem for the noncommutative coordinate rings of affine open subsets of such of those varieties which are smooth (Theorem 3.4). The theorem describes the local ring of a point as a truncation of a quantization of the enveloping Poisson algebra of a smooth commutative local algebra. An explicit description of this quantization is given in Theorem 2.5. A description of the  $A$ -module structure of the Poisson envelope of a smooth commutative algebra  $A$  was given in *loc. cit.*, Theorem 4.1.3. However the proof given in *loc. cit.* has a gap. We fix this gap for  $A$  local (Theorem 1.4) and prove a weaker global result (Theorem 1.6).

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## 0. INTRODUCTION

We consider associative, unital algebras over a fixed algebraically closed field  $k$  of characteristic zero. If  $R$  is an algebra, then the *commutator filtration* of  $R$  is defined as

$$(1) \quad F_0 R = R, \quad F_{n+1} R := \sum_{p=1}^n F_p R F_{n+1-p} R + \sum_{p=0}^n \langle [F_p R, F_{n-p} R] \rangle$$

(One checks that this filtration is the same as that called *NC-filtration* in [2].) By definition,  $FR$  is the smallest of all descending filtrations  $\mathcal{G}$  with  $\mathcal{G}_p \mathcal{G}_q \subset \mathcal{G}_{p+q}$ ,  $[\mathcal{G}_p, \mathcal{G}_q] \subset \mathcal{G}_{p+q+1}$ . An algebra  $R$  is *nilcommutative of order  $d$*  if  $F_{d+1} R = 0$ . Thus the nilcommutative algebras of order 0 are just the commutative algebras. We write  $NC_d$  for the category of nilcommutative algebras of order  $d$  and algebra homomorphisms and set  $NC = \cup_{d=0}^{\infty} NC_d$ . An algebra  $R$  is called  *$d$ -formally smooth* if  $R \in NC_d$  and if  $\text{hom}_{NC_d}(R, \_)$  maps surjections with nilpotent kernel to surjections, and is  *$d$ -smooth* if it is  $d$ -formally smooth and if the commutative algebra

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$A = R/F_1R$  is essentially of finite type. For example a 0-smooth algebra is the same thing as a smooth commutative algebra. For the remainder of this section  $A$  will be a fixed smooth commutative algebra. It was shown in [2], 1.6.1 that there exists a tower of surjective homomorphisms:

$$\dots \twoheadrightarrow R_{d+1} \twoheadrightarrow R_d \twoheadrightarrow \dots \twoheadrightarrow R_2 \twoheadrightarrow R_1 \twoheadrightarrow R_0 = A$$

such that  $R_d$  is  $d$ -smooth and  $R_d/F_dR = R_{d-1}$  ( $d \geq 1$ ). Moreover it is shown in *loc. cit.* that such a tower is unique up to noncanonical isomorphism. Kapranov further develops a theory of nilcommutative  $d$ -smooth algebraic varieties based on this ‘affine’ construction. In this paper we focus on the affine part of Kapranov’s work. We study the structure of the algebras  $R_d$  and of their associated graded Poisson algebras  $GR_d := \bigoplus_{n=0}^d F_n R_d / F_{n+1} R_d$ . A characterization of  $GR_d$  was given in [2] 4.2.1. It was shown that there is an isomorphism

$$(2) \quad PA/P_{>d}A \xrightarrow{\sim} GR_d$$

Here  $P : ((\text{Comm})) \rightarrow ((\text{Pois}))$  is left adjoint to the forgetful functor which associates to a Poisson algebra its underlying commutative algebra. The algebra  $PA$  turns out to be graded, and  $P_{>d}A = \bigoplus_{n>d} P_n A$ . The map (2) is canonical, and comes from the adjointness property of  $P$ ; if  $R \in NC_d$  is any algebra with  $G_0R = R/F_1R = A$ , then the adjunction map  $PA \rightarrow GR$  induced by  $A \cong G_0R$  factors through  $PA/P_{>d}A$  obtaining (2). Here we prove a converse of Kapranov’s result. We show that if  $R_d \in NC_d$  is any algebra with  $R_d/F_1R_d = A$  and such that the canonical map (2) is an isomorphism, then  $R_d$  is  $d$ -smooth (Theorem 3.4). This means that to give a  $d$ -smooth algebra  $R$  with  $R/F_1R = A$  is the same thing as to give an associative multiplication

$$(3) \quad \phi = \sum_{r=0}^d \phi_r : PA/P_{>d}A \otimes P/P_{>d}A \rightarrow PA/P_{>d}A$$

with  $\phi_r$  homogeneous of degree  $r$ . For  $(A, \mathcal{M})$  local, we give (Theorem 2.5) a canonical construction which produces an associative product

$$(4) \quad B^X(\hbar) = \sum_{p=0}^{\infty} B_p^X \hbar^p : PA \otimes PA[[\hbar]] \rightarrow PA[[\hbar]]$$

for each regular system of parameters  $X = \{x_1, \dots, x_n\} \subset \mathcal{M}$ , with  $B_p^X$  homogeneous of degree  $p$  and a bidifferential operator of order  $\leq p$ . It turns out that, modulo  $P_{>d}A$ , the evaluated series  $B^X(1)$  is a finite sum, and gives a product  $\phi$  satisfying the requirements of (3). We use this product to give a local characterization of  $R_d$  (Theorem 3.4). The construction of the product (4) uses a local isomorphism of  $A$ -modules

$$(5) \quad PA \cong S^A L_+^A \Omega_A^1 \quad (n \geq 0)$$

Between  $PA$  and the  $A$ -symmetric algebra of the Lie subalgebra

$$[A\Omega^1] \subset [LA\Omega^1] \subset [LA\Omega^1]$$

of the free  $A$ -Lie algebra  $L^A\Omega_A^1$  generated by the module of Kähler differentials. Theorem 4.1.3 of [2] states that there is a global isomorphism as that of (5); there is however a gap in the proof. The gap is explained in section 1 below, where it is also shown how it is fixed for  $A$  local (Theorem 1.4). I do not know whether (5) still holds globally. A weaker version of (5) which holds globally is proved in Theorem 1.6; it establishes that  $PA$  carries a filtration such that the associated graded module is (globally) isomorphic to the right hand side of (5).

When I explained to Kapranov the gap in his proof of (5), and told him the gap could be fixed locally, he suggested that a weaker version along the lines of that presented here (Theorem 1.6) should hold globally. I am thankful to him for this suggestion.

The remainder of this paper is organized as follows. In section 1 we recall in some detail the construction of the Poisson algebra  $PA$ , –which we call the Poisson envelope of  $A$ – explain the gap in Kapranov’s proof of (5), and prove it in the local case (Theorem 1.4). The section ends with the weaker version of (5) which holds globally (Theorem 1.6). Section 2 is devoted to the construction of the product (4) (Theorem 2.5). The results of this section can be seen as the generalization to general local smooth algebras of those obtained by Kapranov for localizations of polynomial rings ([2]§3). Our approach is however different from that of [2]. In *loc. cit.* the Feynmann-Maslov calculus was used to describe the product of elements in the tensor algebra on a finite dimensional vectorspace  $V$  in terms of a specific ordered basis. Instead we use the coordinate free approach of [1], where explicit formulas for this product were obtained for not necessarily finite dimensional vectorspaces  $V$ . The same formulas apply to the quantized product (4). In section 3 we prove (Theorem 3.4) that, for  $B^X$  as in (4), an algebra  $R \in NC_d$  is (i)  $d$ -smooth  $\Leftrightarrow$  (ii)  $A := R/F_1R$  is smooth commutative and (2) is an isomorphism  $\Leftrightarrow$  (iii)  $R$  is locally isomorphic to  $(PA/P_{>d}A, B^X(1))$  for some regular system of parameters  $X$ . Part (1) $\Rightarrow$ (2) of this was proven by Kapranov in [2] 4.2.1; we give a new proof.

## 1. THE POISSON ENVELOPE OF A COMMUTATIVE ALGEBRA

**1.0. Two gradings in the symmetric algebra of a free Lie algebra.** If  $V$  is a vectorspace, we write  $TV$  for the tensor algebra and  $LV \subset TV$  for the Lie subalgebra it generates. For  $V = \bigoplus_{x \in X} kx$  –the free vectorspace on a set  $X$ –  $LV$  is the free Lie algebra on  $X$ . The symmetric algebra  $S\mathfrak{g}$  of any Lie algebra  $\mathfrak{g}$  is viewed as a Poisson algebra via the Poisson bracket  $\{, \}$  induced by the Lie bracket  $[, ]$  of  $\mathfrak{g}$ . For example

$$\text{Pois} V := SLV$$

is a free Poisson algebra. Fix a vectorspace  $V$  and set  $L = LV$ . We have  $L = \bigoplus_{n \geq 0} L_n$ , where

$$(6) \quad L_0 = V, \quad L_{n+1} = [L_0, L_n] \quad (n \geq 0)$$

Note our grading is the usual one –as defined for example in [3] LA, Ch. IV – shifted down one degree. Put

$$(7) \quad L_n = \begin{cases} L_n & \text{if } L_n \subset L_{n-1} \\ L_{n-1} & \text{if } L_n \not\subset L_{n-1} \end{cases} \quad (n \geq 0)$$

This grading induces one in the symmetric algebra  $S = SL$ ; we write  $S_n$  for its homogeneous part of degree  $n$ . Note that  $S_n$  is not the same thing as the  $n$ -th symmetric power  $S^n = S^n L$ . The latter is the homogenous part of degree  $n$  with respect to a different grading, namely that given by

$$(8) \quad |l|^* = 1 \quad \text{if } l \in L$$

Put

$$L_+ = \bigoplus_{n \geq 1} L_n$$

We have  $S_0 L = SV$  and for  $n \geq 1$

$$S_n L = SV \otimes S_n L_+$$

$$(9) \quad S_n L_+ = \bigoplus_{r \geq 1} \bigoplus_{\substack{0 < i_1 < \dots < i_r \\ p_1 i_1 + \dots + p_r i_r = n \\ p_1, \dots, p_r > 0}} S^{p_1} L_{i_1} \otimes \dots \otimes S^{p_r} L_{i_r}$$

**1.1. Poisson ideals.** A *Poisson ideal* in a Poisson algebra  $P$  is a subspace  $I \subset P$  which is an ideal for both the associative and the Lie algebra structures. If  $Y \subset P$  is a subset, then we put  $\langle Y \rangle$  and  $\langle\langle Y \rangle\rangle$  for the smallest ideal and the smallest Poisson ideal containing  $Y$ . By definition  $\langle Y \rangle \subset \langle\langle Y \rangle\rangle$ . In fact  $\langle\langle Y \rangle\rangle$  is generated as an ideal by the elements of  $Y$  and by those of the form

$$\{a_1, \{a_2, \dots, \{a_n, y\} \dots\}\} \quad n \geq 1, \quad a_i \in P, \quad y \in Y, \quad (1 \leq i \leq n)$$

Furthermore, one checks that if  $X \subset P$  generates  $P$  as a Poisson algebra then for

$$(10) \quad g_i(x_1, \dots, x_n; y) := \{x_1, \{x_2, \dots, \{x_i, \{y, \{x_{i+1}, \dots, \{x_{n-1}, x_n\} \dots\}\}\} \dots\}\}$$

we have

$$(11) \quad \langle\langle Y \rangle\rangle = \langle Y \cup \bigcup_{n=1}^{\infty} \{g_i(x_1, \dots, x_n; y) : 0 \leq i \leq n, \quad x_i \in X, \quad y \in Y\} \rangle$$

**1.2. Poisson envelope.** Let  $A$  be a commutative algebra,  $SA$  the symmetric algebra on its underlying vectorspace,  $SA \twoheadrightarrow A$  the canonical projection,  $IA$  its kernel. The *Poisson envelope* of  $A$  is

$$PA := \frac{SLA}{\langle\langle IA \rangle\rangle}$$

The inclusion  $A = SA/IA \subset PA$  has the following universal property. If  $P$  is a Poisson algebra and  $f : A \rightarrow P$  is a homomorphism of commutative algebras, then there is a unique Poisson homomorphism  $PA \rightarrow P$  which extends  $f$ . In other words  $A \hookrightarrow PA$  is left adjoint to the forgetful functor  $((\text{Pois})) \rightarrow ((\text{Comm}))$  from Poisson

to commutative algebras. One checks that if  $A = SV/I$  is any presentation of  $A$  as a quotient of a symmetric algebra, then  $SLV/\langle\langle I \rangle\rangle$  has the same universal property as and is therefore isomorphic to  $PA$ . In particular

$$PSV = \text{Poiss } V$$

It follows from (11) that if  $I \subset SV$  is as above then  $\langle\langle I \rangle\rangle \subset SLV$  is homogeneous for the grading (7), whence  $PA$  inherits a grading:

$$PA = \bigoplus_{n \geq 0} P_n A$$

For example

$$(12) \quad P_n SV = S_n LV \quad (n \geq 0)$$

In particular

$$(13) \quad P_1 SV = SV \otimes L_1 V = SV \otimes \Lambda^2 V = \Omega_{SV}^2$$

is the module of 2-differential forms. It follows from (13) and (11) that for every commutative algebra  $A$ ,

$$\begin{aligned} P_1 A &= \frac{S_1(LA)}{IAS_1(LA) + \langle\{A, IA\}\rangle} \\ &= \frac{\Omega_{SA}^2}{IA\Omega_{SA}^2 + \Omega_A^1 \wedge dIA} = \Omega_A^2 \end{aligned}$$

Under this isomorphism,

$$\{a, b\} = da \wedge db \in P_1 A$$

For another interpretation of  $P_1 A$  consider the analogy  $L^A$  of the functor  $L$  for  $A$ -modules and  $A$ -Lie algebras. If  $M$  is an  $A$ -module, then  $L^A M$  carries a grading defined exactly as in (6). We have  $L_0^A M = M$ ,  $L_1^A M = \Lambda^2 M$ , and in particular

$$(14) \quad L_1^A \Omega_A^1 = \Omega_A^2 = P_1 A$$

Theorem 4.1.3 of [2] says that a generalization of (14) holds for smooth algebras. Namely it is asserted that for every  $n \geq 0$ ,  $P_n A$  is isomorphic as an  $A$ -module to the homogeneous part of degree  $n$  of the symmetric  $A$ -algebra on  $L^A \Omega_A^1$ :

$$(15) \quad P_n A \cong S_n^A L_+^A \Omega_A^1 \quad (n \geq 0)$$

However the proof of this assertion in [2] has a gap, as the isomorphism given there is not well-defined. Indeed the map in question sends the element

$$P_n A \ni b\{a_0, \{a_1, \dots, \{a_{n-1}, a_n\} \dots\}\} \quad (a_i \in A)$$

to the element

$$b[da_0, [da_1, \dots [da_{n-1}, da_n] \dots]] \in S_n^A L_+^A \Omega_A^1$$

However a calculation shows that this rule maps

$$0 = \{a_1, a_3\{a_2, a_4\}\} + \{a_1, a_2\{a_3, a_4\}\} - \{a_1, \{a_2 a_3, a_4\}\}$$

to the element

$$[da_1, da_3][da_2, da_4] + [da_1, da_2][da_3, da_4]$$

which is nonzero in general. I do not know whether the isomorphism (15) still holds for every smooth algebra  $A$ . It certainly holds for symmetric algebras, as is immediate from (12). We show in Theorem 1.4 below that it also holds for local smooth algebras. For a weaker version of (15) which holds globally, see Theorem 1.6. The following lemma is well-known.

**Lemma 1.3.** *Let  $X$  be a set,  $V = \oplus_{x \in X} kx$  the free vectorspace on  $X$ . Then the set*

$$Y := X \cup \bigcup_{n=1}^{\infty} \{[x_1, [x_2, [\dots, [x_n, x_{n+1}] \dots]] : x_i \in X\}$$

*generates  $LV$  as a vectorspace. In particular there is a basis  $Z$  of  $LV$  such that  $X \subset Z \subset Y$ .*

*Proof.* Straightforward induction.  $\square$

**Theorem 1.4.** *Let  $A$  be a local smooth algebra. Then  $A$  satisfies (15).*

*Proof.* Let  $x_1, \dots, x_n \in A$  be a regular system of parameters and  $V = \oplus_{i=1}^n kdx_i \subset \Omega_A^1$ . We have

$$S^A L_+^A \Omega_A^1 = A \otimes SL_+ V$$

Put  $L = LV$ ; then

$$(16) \quad \Omega_A^1 = A \otimes V, \quad \Omega_{SL_+}^1 = SL_+ \otimes L_+, \quad \Omega_{A \otimes SL_+}^1 = A \otimes SL_+ \otimes L$$

Consider the permutation isomorphism

$$\tau : \Omega_A^1 \otimes SL_+ = A \otimes V \otimes SL_+ \cong A \otimes SL_+ \otimes V$$

Under the identifications (16) the de Rham derivation  $d_{A \otimes SL_+}$  is identified with

$$D := \tau \circ (d_A \otimes id_{SL_+}) + id_A \otimes d_{SL_+}$$

Put

$$\phi = id_{A \otimes SL_+} \otimes [, ] : A \otimes SL_+ \otimes \Lambda^2 L \rightarrow A \otimes SL_+$$

One checks that

$$\{p, q\} := \phi(Dp \wedge Dq)$$

is a Poisson bracket. Note that for  $1 \leq i_1, \dots, i_{r+1} \leq n$ , we have

$$\{x_{i_1}, \{x_{i_2}, \dots, \{x_{i_r}, x_{i_{r+1}}\} \dots\}\} = [dx_{i_1}, [dx_{i_2}, \dots, [dx_{i_r}, dx_{i_{r+1}}] \dots]]$$

It suffices to show that the Poisson algebra  $(A \otimes SL_+, \{, \})$  together with the inclusion  $A = A \otimes k = A \otimes S_0 L_+ \subset A \otimes SL_+$  has the universal property of  $PA$ . Let  $P$  be a Poisson algebra and  $f : A \rightarrow P$  a homomorphism of commutative algebras. Write  $p_i = f(x_i)$ . By lemma 1.3, we may extend  $B_0 = \{dx_1, \dots, dx_n\}$  to a homogeneous basis  $B$  of  $L$  such that every element of  $B' := B \setminus B_0$  be of the form  $[dx_{i_1}, [dx_{i_2}, \dots, [dx_{i_r}, dx_{i_{r+1}}] \dots]]$  ( $r \geq 1$ ). View  $P$  as an  $A$ -module via  $f$  and consider the  $A$ -module homomorphism  $\theta : L_+^A \Omega_A^1 = A \otimes L_+ \rightarrow P$  defined on elements of  $B'$  by

$$(17) \quad \theta[dx_{i_1}, [dx_{i_2}, \dots, [dx_{i_r}, dx_{i_{r+1}}] \dots]] = \{p_{i_1}, \{p_{i_2}, \dots, \{p_{i_r}, p_{i_{r+1}}\} \dots\}\}$$

Note that, as defined,

$$(18) \quad \theta[l_1, l_2] = \{\theta l_1, \theta l_2\} \quad (l_1, l_2 \in B')$$

Indeed by lemma 1.3, the two sides of this identity are defined by the same linear combination of the elements of  $B'$  and of their images. The prescription (17) together with the prescription that  $\theta$  extend  $f$ , determine a unique map  $\theta : A \otimes SL_+ \rightarrow P$  which satisfies (18) for  $l_1, l_2 \in B'' := \{x_1, \dots, x_n\} \cup B'$ . It follows that  $\theta$  is a Poisson homomorphism; uniqueness is clear.  $\square$

**1.5. A weaker version of property (15).** Let  $V$  be a vectorspace. Combining the two gradings (7), (8) we obtain a bigrading

$$(19) \quad SLV = \bigoplus_{p,q \geq 0} S_q^p LV$$

where

$$S_q^p LV := S^p LV \cap S_q LV$$

Now let  $A$  be a commutative algebra, and consider the projection

$$(20) \quad \pi : SLA \twoheadrightarrow PA$$

The map  $\pi$  is homogeneous for the  $\|\ast$ -degree but not for the  $\|\ast$ -degree. However the ideal

$$\mathcal{H}^n := \pi\left(\bigoplus_{p \geq n} S^p LA\right) \quad (n \geq 0)$$

is homogeneous with respect to  $\|\ast$  and therefore the graded ring  $GPA = G_{\mathcal{H}}PA$  is actually bigraded

$$GPA = \bigoplus_{p,q \geq 0} G_q^p PA$$

On the other hand (19) carries over to the free Lie algebra of any  $A$ -bimodule. In particular  $S^A L^A \Omega_A^1$  is a bigraded algebra.

**Theorem 1.6.** *Let  $A$  be a commutative algebra. Then for the bigraded structures defined in 1.5, there is a natural surjective homomorphism of bigraded algebras*

$$\phi : S^A L^A \Omega_A^1 \twoheadrightarrow GPA$$

*If furthermore  $A$  is smooth, then  $\phi$  is an isomorphism.*

*Proof.* The map (20) is the quotient by  $\langle\langle IA \rangle\rangle$ , whence it factors through a map

$$\bar{\pi} : A \otimes SL_+ A = \frac{SLA}{\langle\langle IA \rangle\rangle} \twoheadrightarrow PA$$

In particular  $\bar{\pi}$  induces a homogeneous, surjective homomorphism of graded  $A$ -modules

$$p : A \otimes L_+ A \xrightarrow{\bar{\pi}} \mathcal{H}^1 \twoheadrightarrow G_*^1 PA$$

Write  $\rho : A \rightarrow SA$  for the canonical inclusion. The ideal  $IA$  is generated by the elements

$$u(a, b) := \rho(ab) - \rho a \rho b \quad (a, b \in A)$$

Let  $h_i(a_1, \dots, a_n; b, c)$  be the homogeneous part of  $\|\ast$ -degree one of the element  $g_i(\rho a_1, \dots, \rho a_n; w(b, c))$  of (10). By (11), the elements  $h_i(a_1, \dots, a_n; b, c)$ ,  $1 \leq i \leq n$ ,  $a_i, b, c \in A$  generate  $\ker p$  as an  $A$ -module. A calculation shows that

$$\begin{aligned} h_i(a_1, \dots, a_n; b, c) = & \\ & 1 \otimes \{\rho a_1, \dots, \{\rho a_i, \{\rho(bc), \{\rho a_{i+1}, \dots, \{\rho a_{n-1}, \rho a_n\} \dots\}\} \dots\} - \\ & \rho b \otimes \{\rho a_1, \dots, \{\rho a_i, \{\rho c, \{\rho a_{i+1}, \dots, \{\rho a_{n-1}, \rho a_n\} \dots\}\} \dots\} - \\ & \rho c \otimes \{\rho a_1, \dots, \{\rho a_i, \{\rho b, \{\rho a_{i+1}, \dots, \{\rho a_{n-1}, \rho a_n\} \dots\}\} \dots\} \} \end{aligned}$$

It follows from this that

$$M := \ker(A \otimes A \twoheadrightarrow \Omega_A^1) \oplus \ker p$$

is the Lie ideal generated by  $\ker(A \otimes A \twoheadrightarrow \Omega_A^1)$  in the  $A$ -Lie algebra  $A \otimes LA$ . Hence  $(A \otimes LA)/M = L^A \Omega_A^1$ , and

$$(21) \quad L_+^A \Omega_A^1 \cong G^1 PA$$

The map  $\phi$  of the theorem is that induced by (21); it is surjective because  $G^1 PA$  generates  $GPA$  as an  $A$ -algebra. Assume now that  $A$  is smooth; we must prove that  $\phi$  is injective. This is a local question, so we may further assume that  $A$  is local. Let  $x_1, \dots, x_n$  be a regular system of parameters. By the proof of 1.4 there is an isomorphism  $\psi : PA \xrightarrow{\sim} S^A L_+^A \Omega_A^1$  such that

$$(22) \quad \psi\{x_{i_1}, \{x_{i_2}, \dots, \{x_{i_r}, x_{i_{r+1}}\} \dots\}\} = [dx_{i_1}, [dx_{i_2}, \dots, [dx_{i_r}, dx_{i_{r+1}}] \dots]]$$

Furthermore the induced map  $G\psi : GPA \rightarrow S^A L_+^A \Omega_A^1$  still verifies (22). Thus  $G\psi \circ \phi$  is the identity map, because it is so on generators. In particular,  $\phi$  is injective.  $\square$

## 2. LOCAL QUANTIZATION OF THE POISSON ENVELOPE

**2.0. PBW quantization.** Let  $\mathfrak{g}$  be a Lie algebra,  $S\mathfrak{g}$  and  $U\mathfrak{g}$  the symmetric and universal enveloping algebras, and consider the symmetrization map

$$(23) \quad e : S\mathfrak{g} \rightarrow U\mathfrak{g} \quad e(g_1 \dots g_n) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} g_{\sigma(1)} \dots g_{\sigma(n)}$$

By the Poincaré-Birkhoff-Witt theorem, the associative product

$$B : S\mathfrak{g} \otimes S\mathfrak{g} \rightarrow S\mathfrak{g}, \quad B(x \otimes y) = e^{-1}(exey)$$

decomposes as a sum

$$(24) \quad B = \sum_{p=0}^{\infty} B_p \quad \text{where} \quad B_p(S^n \mathfrak{g}) \subset S^{n-p} \mathfrak{g} \quad (n, p \geq 0)$$

We have  $B_0(x \otimes y) = xy$ ,  $B_1(x \otimes y) = \frac{1}{2}\{x, y\}$ . Explicit formulas for all the  $B_p$  are given in [1]. It also proved in *loc. cit.* that for each  $p \geq 0$ ,  $B_p$  is a differential operator of order  $\leq p$ . We call the map

$$B(\hbar) := \sum_{n \geq 0} B_n \hbar^n : S\mathfrak{g} \otimes S\mathfrak{g}[[\hbar]] \rightarrow S\mathfrak{g}[[\hbar]]$$

the PBW *quantization*. The next lemma establishes the properties of the product  $B$  with respect to the commutator filtration (1) in the case when  $\mathfrak{g}$  is free.



**Lemma 2.1.** *Let  $V$  be a vectorspace,  $LV$  the free Lie algebra,  $TV$  the tensor algebra, and  $e$  as in (23). Then*

- (1)  $e(\text{Poiss}_{\geq n} V) = F_n TV$
- (2) *The operator  $B_p : \text{Poiss } V \otimes \text{Poiss } V \rightarrow \text{Poiss } V$  of (24) is homogeneous of degree  $+p$  for the  $\|\cdot\|_*$ -degree (7).*

*Proof.* One checks that if  $\mathfrak{g}$  is any Lie algebra and  $U\mathfrak{g}$  its enveloping algebra, then for  $F_0\mathfrak{g} = \mathfrak{g}$ ,  $F_d\mathfrak{g} = [\mathfrak{g}, F_{d-1}\mathfrak{g}]$  ( $d \geq 1$ ) we have

$$F_n U\mathfrak{g} = \sum_r \sum_{d_1 + \dots + d_r \geq n} F_{d_1}\mathfrak{g} \cdot \dots \cdot F_{d_r}\mathfrak{g}$$

For  $\mathfrak{g} = LV$ , we obtain

$$(25) \quad F_n TV = \sum_r \sum_{d_1 + \dots + d_r \geq n} L_{d_1} V \cdot \dots \cdot L_{d_r} V$$

From (25) and (9) it is clear that  $e(\text{Poiss}_{\geq n} V) \subset F_n TV$ . We must show that  $F_n TV \subset e(\text{Poiss}_{\geq n} V)$ . Consider the following subspace of  $F^n TV$

$$\mathcal{A}_{p,n} := \sum_{r \leq p} \sum_{d_1 + \dots + d_r \geq n} L_{d_1} V \cdot \dots \cdot L_{d_r} V$$

Clearly  $F_n TV = \cup_{p \geq 1} \mathcal{A}_{p,n}$ . An inductive argument similar to that of the usual proof of the surjectivity of  $e$  shows that for  $p \geq 1$ ,  $\mathcal{A}_{p,n} \subset e(\text{Poiss}_{\geq n} V)$ . This proves assertion (1). Assertion (2) follows by counting degrees in the formula for  $B_p$  given in [1] 1.1.  $\square$

**Corollary 2.2.** *(Compare [2], 3.4.7) The natural map*

$$\text{Poiss } V = PSV \xrightarrow{\sim} \bigoplus_{n=0}^{\infty} F_n TV / F_{n+1} TV$$

*is an isomorphism.*  $\square$

**2.3. PBW quantization of  $PA$  for  $A$  local and smooth.** Let  $(A, \mathcal{M})$  be a smooth local commutative algebra and  $X = \{x_1, \dots, x_n\} \subset \mathcal{M}$  a regular system of parameters. Set  $V = k^n$ . We are going to combine Theorem 1.4 and the PBW quantization of  $\text{Poiss } V$  to obtain an associative product

$$B^X(\hbar) = \sum_{p=0}^{\infty} B_p^X \hbar^p : PA \otimes PA[[\hbar]] \rightarrow PA[[\hbar]]$$

Because the map  $B_p : \text{Poiss } V \otimes \text{Poiss } V \rightarrow \text{Poiss } V$  is a differential operator, it is continuous with respect to the topology of any ideal  $I \subset \text{Poiss } V$ . Applying this for  $I = V \cdot \text{Poiss } V$  and completing we obtain the horizontal solid arrow in the following commutative diagram

$$(26) \quad \begin{array}{ccc} PA \otimes PA & \xrightarrow{B_p^X} & PA \\ \downarrow \iota_X \hat{\otimes} \iota_X & & \downarrow \iota_X \\ k[[t]] \hat{\otimes} SL_+ V \hat{\otimes} k[[t]] \hat{\otimes} SL_+ V & \xrightarrow{\hat{B}} & k[[t]] \hat{\otimes} SL_+ V \end{array}$$

Here  $\hat{\otimes}$  is the completed tensor product and  $k[[t]]$  is shorthand for  $k[[t_1, \dots, t_n]]$ . The map  $\iota_X$  is the composite

$$\iota_X : PA \xrightarrow{\alpha_X} A \otimes SL_+ V \hookrightarrow \hat{A} \hat{\otimes} SL_+ V \xrightarrow{j_X \otimes 1} k[[t]] \hat{\otimes} SL_+ V$$

where  $\alpha_X$  is the isomorphism of 1.4,  $\hookrightarrow$  is the passage to completion and  $j_X : \hat{A} \cong k[[t]]$  is the isomorphism determined by  $x_i \mapsto t_i$  ( $i = 1, \dots, n$ ). Because  $A \hookrightarrow \hat{A}$  is injective, so are both vertical maps in (26). The map  $B_p^X$  is defined by the following lemma.

**Lemma 2.4.** *The map  $\hat{B}_p$  of (26) sends the image of  $\iota_X \hat{\otimes} \iota_X$  to the image of  $\iota_X$ .*

*Proof.* Let  $Z = X \cup \{y_1, y_2, \dots\}$  be a basis as that of Lemma 1.3. Every monomial on the elements of  $Z$  can be written as  $x^{\alpha'} y^{\alpha''} = \prod_{i=1}^n x_i^{\alpha'(i)} \prod_{j=1}^{\infty} y_j^{\alpha''(j)}$  for some multi-indices  $\alpha' : \{1, \dots, n\} \rightarrow \mathbb{Z}_{\geq 0}$  and  $\alpha'' : \mathbb{Z}_{\geq 1} \rightarrow \mathbb{Z}_{\geq 0}$  with  $\alpha''(n) = 0$  for  $n \gg 0$ . Let  $\frac{\partial^{|\alpha|}}{\partial x^{\alpha'} \partial y^{\alpha''}}$  be the higher derivation with symbol  $x^{\alpha'} y^{\alpha''}$ . Because  $B_p : SLV \otimes SLV \rightarrow SLV$  is a bidifferential operator of order  $\leq p$ , it can be written as an  $SLV$ -linear combination of cup products of higher derivations with respect to the basis  $Z$

$$B_p = \sum_{|\alpha|, |\beta| \leq p} c_{\alpha, \beta} \frac{\partial^{|\alpha|}}{\partial x^{\alpha'} \partial y^{\alpha''}} \cup \frac{\partial^{|\beta|}}{\partial x^{\beta'} \partial y^{\beta''}}$$

Since each of the higher derivations above maps  $A \subset k[[t]]$  to itself, so does  $\hat{B}_p$ .  $\square$

**Theorem 2.5.** *Let  $(A, \mathcal{M})$  be a smooth local algebra,  $X \subset \mathcal{M} \cdot A_{\mathcal{M}}$  a regular system of parameters,  $p \geq 0$ ,  $B_p^X$  as in (26) and  $\widehat{PA} = \varprojlim_d PA/P_{>d}A = \prod_{d=0}^{\infty} P_d A$ . Then*

- (1)  $B^X(\hbar) = \sum_{p=0}^{\infty} B_p^X \hbar^p : PA \otimes PA[[\hbar]] \rightarrow PA[[\hbar]]$  is associative.
- (2)  $B_p^X$  is a differential operator of order  $\leq p$ .
- (3)  $B_p^X(P_n A \otimes P_m A) \subset P_{n+m+p} A$ .
- (4) The map  $B^X(\hbar)$  induces a continuous associative product

$$B^X(1) := \sum_{p=0}^{\infty} B_p^X : \widehat{PA} \hat{\otimes} \widehat{PA} \rightarrow \widehat{PA}$$

- (5) For the associative algebra  $Q_X = (\widehat{PA}, B^X(1))$ , we have

$$F_n Q_X = \prod_{d \geq n} P_d A$$

- (6) There is a commutative diagram of monomorphisms

$$\begin{array}{ccc} k\{t_1, \dots, t_n\} & \xrightarrow{\quad} & Q_X \\ & \searrow \text{inc} & \downarrow \\ & & k\{\{t_1, \dots, t_n\}\} \end{array}$$

where  $\text{inc}$  is the canonical inclusion of the noncommutative polynomials into the noncommutative power series

*Proof.* Assertions (1), (2) and (3) are immediate from the analogous properties of the PBW quantization; (4) follows from (3), and (5) from Lemma 2.1. It is clear from the definition of  $Q_X$  that there is a diagram as that in (5) but with  $\widehat{SLV} := (\prod_{n \geq 0} k[[t]] \hat{\otimes} SL_+ V, \hat{B})$  substituted for  $k\{\{t\}\}$ . Note that, for  $I = \sum t_i \cdot k[t_1, \dots, t_n]$ ,  $\widehat{SLV}$  is the completion of  $SLV$  with respect to the filtration  $\{I \cdot SLV + S_{\geq n} LV : n \geq 0\}$ , and that  $J := e^{-1}(\langle t_1, \dots, t_n \rangle) = I \oplus S_+ LV$ . By lemmas 2.1 and 2.6  $\widehat{SLV} \cong \varprojlim_n TV/e(J)^n = k\{\{t_1, \dots, t_n\}\}$ .  $\square$

**Lemma 2.6.** *Let  $G = \oplus_{n=0}^{\infty} G_n$  be a graded commutative algebra. Assume  $G$  is additionally equipped with an associative –but not necessarily commutative– product*

$$\Phi = \sum_{p=0}^{\infty} \Phi_p : G \otimes G \rightarrow G$$

*such that  $\Phi_0$  is the original commutative product, and that for each  $p \geq 1$ ,  $\Phi_p$  is a bidifferential operator. Let  $I \subset G_0$  be an ideal for  $\Phi_0$ , and  $J \subset G$  the  $\Phi$ -ideal it generates. Then the linear topologies induced on the underlying vectorspace of  $G$  by the filtrations  $\{J^n + G_{\geq n} : n \geq 0\}$  and  $\{I^n G + G_{\geq n} : n \geq 0\}$  coincide.*

*Proof.* It suffices to prove that for each  $d \geq 0$  the filtrations  $\{J^n + G_{\geq d+1}/G_{\geq d+1} : n \geq 0\}$  and  $\{I^n G + G_{\geq d+1}/G_{\geq d+1} : n \geq 0\}$  of  $G/G_{\geq d+1}$  are equivalent. Thus we may assume that  $G_m = 0$  for  $m \geq d+1$ . Hence  $\Phi$  is a bidifferential operator. Let  $\alpha$  be the order of  $\Phi$ . We write  $x \star y := \Phi(x \otimes y)$ , and if  $X \subset G$  is any subspace, we put  $X^{\star n}$  for the subspace generated by all products  $x_1 \star \dots \star x_n$  with  $x_i \in X$ . Let  $i \in I$ ,  $p \geq 1$ ,  $n, r \geq 0$ ,  $F_i(x) := i \star x$ . Because  $F_i$  is a differential operator of order  $\leq \alpha$ ,

$$(27) \quad F_i(I^{p\alpha+r} G_n) \subset I^{p\alpha+r+1} G_n + I^p G_{\geq n+1}$$

Using (27) one checks by induction that for  $(c_r \dots c_0) := \sum_{i=0}^r c_i \alpha^i$ ,

$$(28) \quad I^{\star(c_r \dots c_0)} \subset M_{(c_r \dots c_0)} := \sum_{j=0}^r I^{(c_r \dots c_j)} G_{\geq j} + G_{\geq r+1}$$

We remark that  $M_{(c_r \dots c_0)}$  is an ideal for both  $\star$  and the original product. Hence for  $N \geq (c_r \dots c_0) + d$  and  $r \geq d$ ,

$$J^{\star N} \subset (I \oplus G_{\geq 1})^{\star N} \subset I^{\star(c_r \dots c_0)} \star G_{\geq d} \subset M_{(c_r \dots c_0)} \subset I^{(c_r \dots c_d)} G$$

where the subindex  $\star$  denotes two sided ideal generated by the product  $\star$ . Now using (28), and noting that  $I^{\star n} \star G_d = I^n G_d$  and that in general the projection  $G \twoheadrightarrow G_j$  maps  $I^{\star n} \star G_j$  surjectively onto  $I^n G_j$ , one checks that, for  $r \geq d$

$$M_{(c_r \dots c_0)} = \sum_{j=0}^d I^{\star(c_r \dots c_j)} \star G_{d-j}$$

It follows that

$$J^{\star n} \supset M_{n\alpha^d} \supset I^{n\alpha^d} \quad \square$$

## 3. SMOOTH NILCOMMUTATIVE AND NIL-POISSON ALGEBRAS

**3.0 Nil-Poisson algebras.** Let  $P$  be a Poisson algebra. Put  $F_0P = P$  and inductively

$$(29) \quad F_{n+1}P := \sum_{i=1}^n F_i P F_{n+1-i} P + \sum_{i=0}^n \langle \{F_i P, F_{n-i} P\} \rangle$$

for  $n \geq 0$ . This is the Poisson analogue of the commutator filtration (1). For example if  $A$  is any commutative algebra then

$$F_r P A = P_{\geq r} A = \bigoplus_{n=r}^{\infty} P_n A \quad (r \geq 0)$$

The analogue of a nilcommutative algebra of order  $\leq d$  is called a *nil-Poisson* algebra of order  $\leq d$ . The category of ni-Poisson algebras of order  $\leq d$  is  $NP_d$ . We put  $NP = \cup_{d \geq 0} NP_d$ . Formal  $d$ -smoothness and  $d$ -smoothness for objects of  $NP_d$  are the obvious analogues of the same properties for objects of  $NC_d$  as defined in §1. If  $A$  is (formally) smooth in the commutative sense, then  $PA/P_{>d}A$  is (formally)  $d$ -smooth. It turns out that every (formally)  $d$ -smooth Poisson algebra is of this form; see proposition 3.3. The following Lemma is the analogue of [2], 1.2.7 for Poisson algebras.

**Lemma 3.1.** *Let  $P \in NP_d$  and  $f : P \rightarrow P$  a Poisson endomorphism. Assume the induced map  $P/F_1P \rightarrow P/F_1P$  is the identity. Then the restriction of  $f$  to  $F_{d+1}P$  is the identity also.*

*Proof.* Consider the map  $D : P \rightarrow P$ ,  $Dp := fp - p$ . We have

$$(30) \quad \begin{aligned} D\{p, q\} &= \{Dp, q\} + \{p, Dq\} - \{Dp, Dq\} \\ D(pq) &= pDq + qDp - DpDq \end{aligned}$$

By hypothesis,  $DP \subset F_1P$ ; it follows from this, using (30), (29) and induction, that for  $n \geq 0$ ,  $D(F_nP) \subset F_{n+1}P$ . In particular  $D(F_dP) = 0$ .  $\square$

*Remark 3.2.* The proof of the lemma above still applies if one substitutes  $NC_d$  for  $NP_d$  and “algebra endomorphism” for “Poisson endomorphism”. This gives an alternate proof of [2], 1.2.7.

**Proposition 3.3.** *Let  $P \in NP_d$ ,  $A = P/F_1P$ . Then the following conditions are equivalent*

- i)  $P$  is  $d$ -formally smooth.
- ii)  $A$  is 0-formally smooth and  $PA/P_{>d}A \cong P$ .

*The same holds if we replace “formally smooth” by “smooth” in both i) and ii).*

*Proof.* That ii)  $\Rightarrow$  i) is clear, as is that i) implies  $A$  is formally smooth. Use the formal smoothness of  $A$  to obtain a section  $s : A \rightarrow P \in ((\text{Comm}))$  of the projection  $P \twoheadrightarrow A$ , and then the universal property of  $PA$  to lift  $s$  to a map of extensions  $\alpha : PA/P_{>d}A \rightarrow P$ . To prove that if i) holds then  $\alpha : PA/P_{>d}A \rightarrow P$  is an

isomorphism, note that, by the hypothesis on  $P$ , there is a map  $\beta : P \rightarrow PA/P_{>d}A$  which descends to the identity of  $A$ . One is thus reduced to showing that  $\alpha\beta$  and  $\beta\alpha$  are isomorphisms. This follows from lemma 3.1.  $\square$

Part (1) $\Rightarrow$ (2) of the following theorem is due to Kapranov ([2] 4.2.1); we give a new proof.

**Theorem 3.4.** *Let  $R \in NC_d$ ,  $G = \bigoplus_{n=0}^d G_n$  the associated graded Poisson algebra,  $A = G_0$ ,  $\pi : R \rightarrow A$  the projection. The following conditions are equivalent*

- (1)  *$R$  is  $d$ -smooth.*
- (2)  *$A$  is 0-smooth and the canonical map  $PA/P_{>d}A \rightarrow G$  is an isomorphism.*
- (3) *For every maximal ideal  $\mathcal{M} \subset A$ , there is a regular system of parameters  $X \subset \mathcal{M} \cdot A_{\mathcal{M}}$  such that for  $Q_X$  as in theorem 2.5, the Ore localization of  $R$  at  $\pi^{-1}(\mathcal{M})$  is isomorphic to  $Q_X/F_{>d}Q_X$ .*

*Proof.* Assume (1) holds. Then clearly  $A$  is 0-smooth. Let  $\mathcal{M} \subset A$  be a maximal ideal,  $\mathbf{M} = \pi^{-1}(\mathcal{M})$ ,  $X \subset \mathcal{M} \cdot A_{\mathcal{M}}$  a regular system of parameters and  $Q = Q_X/F_{>d}Q_X$ . Because the Ore localization  $R_{\mathbf{M}}$  is  $d$ -smooth, the identity of  $A_{\mathcal{M}}$  can be lifted to a map  $f : R_{\mathbf{M}} \rightarrow Q$ . By lemma 3.1, the map induced by  $f$  at the graded level is an isomorphism, whence  $f$  is an isomorphism. We have just proved that (1) $\Rightarrow$ (3). It is clear that (3) $\Rightarrow$ (2). We prove next that (2) $\Rightarrow$ (1), by induction on  $d \geq 0$ . The case  $d = 0$  is tautological. Assume  $d \geq 1$  and that the theorem is true for  $d - 1$ . Let  $R \in NC_d$  satisfy the hypothesis of the theorem. Then by inductive assumption  $R_{d-1} = R/F_dR$  is  $d - 1$ -smooth. Let  $\pi : U \rightarrow R_{d-1}$  be the universal central extension as defined in [2] 1.3.6. By [2] 1.6.2,  $U$  is  $d$ -smooth. One checks that  $\ker \pi = F_dU$ . By [2] 1.3.8, there is a map  $\alpha : U \rightarrow R$  which induces the identity of  $R_{d-1}$ . Consider the Poisson homomorphism  $\beta$  induced by  $\alpha$  at the associated graded level. Because (1) $\Rightarrow$ (2),  $\beta$  is an endomorphism of  $PA/P_{>d}A$ . By virtue of Lemma 3.1, because  $\beta$  induces the identity modulo  $P_dA$  and is homogeneous, it has to be the identity. Thus  $\alpha$  is an isomorphism.  $\square$

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